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Symmetry-breaking toroidal perturbations of a stellarator magnetic field

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Abstract. Analytic expressions for the magnetic field of a toroidal stellarator produced by a set of finite-size helical conductors wound on a torus have been obtained by a direct application of the Biot–Savart law. The effects of toroidal curvature have been included as toroidal perturbations which break the helical symmetry of a corresponding straight system. On expressing the problem of field line behaviour as the problem of a conservative Hamiltonian dynamical system under small non-stationary perturbations, approximate toroidal magnetic surfaces have been obtained by an asymptotic method. Numerical examples for simple cases are given to illustrate the effects of toroidal curvature.

1. Introduction

Calculations of stellarator magnetic fields in toroidal geometry are important (*a*) for plasma containment in stellarators and torsatrons and (*b*) for stabilisation of MHD modes by means of vacuum rotational transforms produced by external helical coils in tokamaks (Hicks *et al* 1980) and in high-β diffuse pinches (Ohkawa *et al* 1980). Analysis of toroidal stellarator magnetic fields has been carried out by numerous authors (see, for example, Miyamoto 1978, and references therein). However, there are important reasons to motivate yet another perturbative theory which is the main concern of this paper.

It is well known that the magnetic field line equations of a cylindrical stellarator possess an exact invariant which may be identified as the flux function of magnetic surfaces. The existence of this invariant is a direct consequence of helical symmetry in a straight system and this is independent of the amplitude or the pitch of the helical field. Moreover, the 'motion' of the field lines may be described by an integrable 'Hamiltonian' system (Filonenko *et al* 1967). Much of our intuition on stellarator magnetic fields has been developed from detailed studies of this case (Morozov and Solov'ev 1966). On the other hand, the absence of any symmetry, other than axial symmetry, in toroidal geometry has rendered the study of toroidal stellarator magnetic fields much more difficult and rigorous proofs of the existence of exact magnetic surfaces for this case have not appeared for this reason (Grad 1967). The bending and joining of periodic ends of a cylindrical stellarator to form a toroidal one introduce a toroidal curvature which breaks the helical symmetry of the straight system. In general, magnetic surfaces for this case can be expected to exist only in an approximate sense. This expectation has been supported by numerical calculations (Gibson 1967,

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Blamey *et al* 1982) which, unfortunately, do not yield any general quantitative estimates on the extent to which a given field line may be said to be confined to a magnetic surface. This question may be considered as a problem in the study of dynamical systems close to integrable ones (e.g. Nehoroshev 1977). The purpose of this paper is two-fold. (1) To obtain the symmetry-breaking toroidal perturbations for the class of stellarator magnetic fields which have cylindrical symmetry in the zeroth order; and (2) to formulate the problem as a problem of a conservative Hamiltonian system under small non-stationary perturbations and thus take advantage of established methods to obtain some consequences of toroidal curvature.

The perturbation theory developed here is intended to be sufficiently flexible to be applicable to a variety of current distributions and helical winding laws. In the next section, it is shown how a current distribution produced by a set of helical conductors with rectangular cross sections wound on the surface of a torus may be Fourier analysed into a series of a harmonic current distributions. Such a harmonic analysis will be seen to simplify the problems considered.

The nature of the magnetic field produced depends sensitively on the current distribution on the toroidal surface. Hence it is important to satisfy the boundary conditions which have not always received adequate attention in the literature. This aspect is discussed in § 3 and a general method of solution which ensures that the boundary conditions are satisfied is indicated there.

The present theory employs a direct inverse aspect ratio expansion without resorting to ordering schemes used by previous authors (Greene and Johnson 1961, Dobrott and Frieman 1971). Here, the cylindrical case plays a fundamental role and the conditions under which the cylindrical limit is obtained are derived in § 4. In passing, exact solutions to some cylindrical problems which have hitherto received only approximate treatments are also written down in § 4.

The main results of this paper are presented in § 5, where the effect of toroidal curvature is seen to produce symmetry-breaking toroidal perturbations. The field line equations of the toroidal stellarator are cast in the form of a conservative Hamiltonian system under small non-stationary perturbations which arise from toroidal curvature. The system of equations is discussed using the method of averaging and approximate magnetic surfaces are obtained to indicate some of the consequences of toroidal geometry in § 6. In the concluding section a summary is presented and further developments are indicated.

2. Harmonic analysis of a current distribution

In a previous paper (Sy 1981), it was shown that the use of proper toroidal coordinates can lead to accurate expressions for the magnetic field. Unfortunately, the toroidal perturbations which break the helical symmetry of a cylindrical stellarator cannot be obtained conveniently from these expressions. Instead, it will be convenient to obtain these perturbations, using the common quasi-toroidal coordinates (r, θ, ϕ), where θ and ϕ are respectively poloidal and toroidal angles. These coordinates have the additional advantage of simple geometric interpretation inside a torus, which will be taken to have minor radius a and major radius R_0.

In practice, current-carrying helical conductors usually come in the form of copper strips with a rectangular cross section having a width D and thickness T. To produce the classical stellarator magnetic field, a system of such conductors carrying a uniform current is wrapped on the toroidal surface (see figure 1) according to the classical winding law

$$\theta - \theta_0 = (m/l)\phi \tag{1}$$

where θ_0 is the initial poloidal angle at the $\phi = 0$ meridonal plane, m is the number of toroidal periods and l is the multiplicity of the device. Although the winding law (1) may be generalised to include modulations, discussions are here limited to this form.

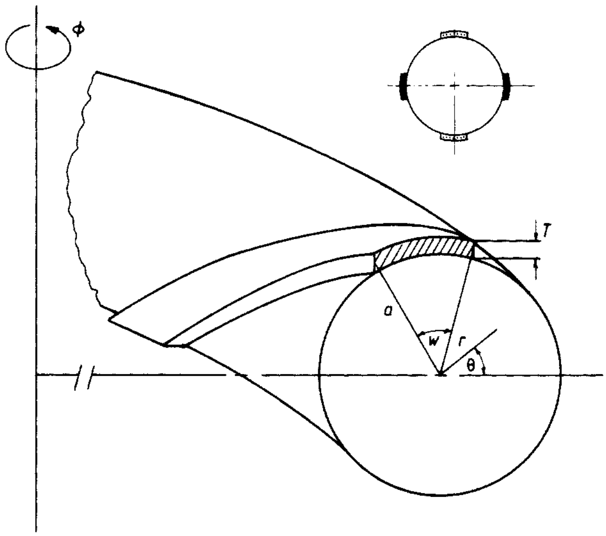

Figure 1. Schematic definitions of the angular width w of the helical conductors and of the quasi-toroidal coordinates (r, θ, ϕ). The insert indicates the $\phi = 0$ locations of helical conductors of a classical $l = 2$ stellarator.

The angle w (radians) subtended at the minor axis by the helical conductor is given by

$$w = D/a \cos\sigma \tag{2}$$

where σ is the angle between the conductor and the generatrix in the toroidal direction. From geometric considerations, toroidal effect gives rise to a modulated $\cos\sigma$, which reads, to leading terms,

$$\cos\sigma = \frac{1}{1+\nu^2}\left(1 - \frac{1}{2}\frac{\varepsilon\nu^2\cos\theta}{1+\nu^2}\right) \tag{3}$$

where θ is the poloidal position of the centre line of the conductor, $\varepsilon = a/R_0$ is the inverse aspect ratio and $\nu \equiv \varepsilon m/l$. For typical cases of interest, one has $\nu < 1$ and $\varepsilon\nu^2/2(1+\nu^2) \ll 1$. Hence w in (2) may be taken as a constant, independent of θ, to a sufficient degree of accuracy.

If the total current through each conductor is I_a, then the current distribution for the toroidal stellarator may be specified by

$$\mathrm{d}I/\mathrm{d}\theta_0 = (I_a/w)f(\theta_0) \tag{4}$$

where $f(\theta_0)$ is a series of step functions, specifying the locations of the finite-size conductors and directions of their currents. In the case of a stellarator of multiplicity l

$$f(\theta_0)=\begin{cases}1 & 0\leqslant\theta_0\leqslant w/2 \qquad 2\pi-w/2\leqslant\theta_0\leqslant 2\pi\\ (-1)^n & n\pi/l-w/2\leqslant\theta_0\leqslant n\pi/l+w/2\\ 0 & \text{otherwise}\end{cases}\tag{5}$$

where $n=1,2,3,\ldots,2l-1$. The corresponding torsatron has half the number of windings with currents all flowing in the same direction. In this case, $f(\theta_0)$ is still formally given by (5), but values of n are restricted to $2,4,\ldots,2l-2$. Fourier analysis of $f(\theta_0)$ shows that for stellarators

$$\frac{\mathrm{d}I}{\mathrm{d}\theta_0}=\frac{2lI_a}{\pi}\sum_n a_n\cos n\theta_0\tag{6}$$

where the summation is taken over $n=l, 3l, 5l,\ldots$, whilst for torsatrons

$$\frac{\mathrm{d}I}{\mathrm{d}\theta_0}=\frac{lI_a}{\pi}\left(\frac{1}{2}+\sum_n a_n\cos n\theta_0\right)\tag{7}$$

with summation over $n=l, 2l, 3l,\ldots$. For either case, the Fourier coefficients in these expressions are given by

$$a_n=\frac{\sin(nw/2)}{(nw/2)}.\tag{8}$$

The above results show that, in general, torsatron fields have all harmonics of l, whilst stellarator fields have only the odd harmonics. Certain harmonics may be eliminated by choosing a width w, such that $\sin(nw/2)=0$. The case of infinitesimally thin filamentary conductors may be obtained in the limit as w vanishes. Magnetic fields due to such singular current distributions for a cylindrical stellarator has been considered by a number of authors (Aleksin 1962, Maschke 1969). It may be seen from (8) that the Fourier coefficients for this case are all equal to unity, i.e. $a_n=1$ for all n, and hence the harmonics all have equal strengths. It will be seen in § 4 that the difficulties in obtaining exact magnetic surfaces by the previous authors can be overcome by the present method of harmonic analysis.

So far, the discussion on the analysis of the current distribution has not taken into account the finite radial thickness T of the helical conductors. Provided $T^2\ll a^2w$ (this will be shown in § 4) the radial thickness has little consequence and more importantly, it does not give rise to any additional harmonics.

3. Magnetic vector potential and magnetic field

In principle, any vacuum magnetic field may be derived from a suitable magnetic scalar potential, which satisfies Laplace's equation. Since formal solutions to Laplace's equation are known in toroidal coordinates (see e.g. Moon and Spencer 1971) it may appear that a particular potential may be obtained by matching fields at current layers on the toroidal surface (Kovrizhnykh 1963). However, it has been pointed out by Bhadra (1968) that this method does not generally simplify the problem since for realistic current distributions, such as those considered in the previous section, the boundary conditions lead to difficult problems in the determination of the coefficients of the toroidal harmonics.

Since the Biot–Savart law is actually a formal solution to the general boundary-value problem by the use of a Green function, its use will guarantee satisfaction of the boundary conditions though, in practice, difficult integration problems may be encountered. In this paper, the method of the Biot–Savart law is used and it will be shown that the integration problems may be solved in a systematic way.

On account of the toroidal geometry of the current distribution, it is convenient to use the form of the Biot–Savart law derived for quasi-toroidal coordinates (Sy 1981). In contrast to a previous application of this method (Aleksin 1963), advantage is taken of the symmetry of a stellarator current distribution before integrals in the Biot–Savart law are evaluated. This leads to significant simplifications.

It follows from the previous section that the current distribution may be Fourier analysed into a sum of harmonic current distributions. It will suffice mathematically to discuss one harmonic distribution, which will be taken hereafter to be the fundamental, since the complete solution may be obtained by superposition of the various harmonics. Consider then a stellarator magnetic field produced by the harmonic current distribution

$$\mathrm{d}I/\mathrm{d}\theta_0 = I_0 \cos l\theta_0 \tag{9}$$

where, from (6), $I_0 = 2lI_a a_l/\pi$.

3.1. *Toroidal magnetic vector potential*

The toroidal magnetic vector potential may be expressed (Sy 1981) as a double integral over the current distribution (9) on the torus. The components read (SI units)

$$A_\alpha = (\mu_0 I_0/4\pi) \iint_0^{2\pi} (a_\alpha/d) \cos(l\theta' - m\phi')\, \mathrm{d}\theta'\, \mathrm{d}\phi' \tag{10}$$

where the Euclidean distance function d is given by

$$d^2 \equiv 2(1+\varepsilon \cos\theta')(1+\varepsilon\rho\cos\theta)(\mu - \cos\psi) \tag{11}$$

with $\varepsilon = a/R_0$, $\rho \equiv r/a$ and

$$\mu \equiv 1 + \frac{1}{2}\frac{a^2 + r^2 - 2ar\cos(\theta'-\theta)}{(R_0 + a\cos\theta')(R_0 + r\cos\theta)}. \tag{12}$$

The components a_α may be separated for convenience according to powers of the inverse aspect ratio, $a_\alpha = a_\alpha^{(0)} + a_\alpha^{(1)}$. Explicitly, the zeroth-order components read

$$\begin{aligned} a_r^{(0)} &= -\cos\theta\sin\psi + \nu(\sin\theta\cos\theta' - \cos\theta\sin\theta'\cos\psi) \\ a_\theta^{(0)} &= \sin\theta\sin\psi + \nu(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\psi) \\ a_\phi^{(0)} &= \cos\psi - \nu\sin\theta'\sin\psi \end{aligned} \tag{13}$$

whilst the first-order components read

$$a_r^{(1)} = -\varepsilon\cos\theta'\cos\theta\sin\psi \qquad a_\theta^{(1)} = \varepsilon\cos\theta'\sin\theta\sin\psi \qquad a_\phi^{(1)} = \varepsilon\cos\theta'\cos\psi \tag{14}$$

where the symbol $\psi \equiv \phi' - \phi$ has been introduced and $\nu \equiv \varepsilon m/l$ as before. There are no higher-order components for a_α.

3.2. *Toroidal magnetic field*

Similarly, the components of the toroidal magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ which corresponds to (10), read

$$B_\alpha = \frac{(\mu_0 I_0 \varepsilon)}{2\pi a} \iint_0^{2\pi} (b_\alpha/d^3) \cos(l\theta' - m\phi') \, \mathrm{d}\theta' \, \mathrm{d}\phi'. \tag{15}$$

Here, the separation of the components b_α according to powers of the inverse aspect ratio, $b_\alpha = b_\alpha^{(0)} + b_\alpha^{(1)} + \ldots$, is not straightforward. Nevertheless, it may be accomplished from a knowledge of the orders of the ϕ' or equivalently the ψ integrals (see appendix 1). It will be seen that the components of b_α have terms up to and including the third order in the inverse aspect ratio, and no higher orders. These read

$$\begin{aligned} b_r^{(0)} &= -\varepsilon \sin \zeta + \nu \cos \zeta \sin \psi \\ b_\theta^{(0)} &= \varepsilon \cos \zeta + \nu \sin \zeta \sin \psi - \varepsilon\rho \cos \psi \\ b_\phi^{(0)} &= \varepsilon\nu(\rho \cos \zeta - \cos \psi) \end{aligned} \tag{16}$$

$$\begin{aligned} b_r^{(1)} &= -\varepsilon^2 \cos \theta' \sin \zeta + \sin \theta(1 - \cos \psi) + \varepsilon\nu \cos \theta \sin \psi \\ b_\theta^{(1)} &= \varepsilon^2 \cos \theta' \cos \zeta + \cos \theta(1 - \cos \psi) + \varepsilon\nu(\rho \sin \theta' - \sin \theta) \sin \psi \\ b_\phi^{(1)} &= \varepsilon \sin \psi(\rho \sin \theta - \sin \theta') + \nu \cos \theta'(1 - \cos \psi) \end{aligned} \tag{17}$$

$$\begin{aligned} b_r^{(2)} &= \varepsilon \sin(\zeta + 2\theta)(1 - \cos \psi) \\ b_\theta^{(2)} &= \varepsilon \cos(\zeta + 2\theta)(1 - \cos \psi) \end{aligned} \tag{18}$$

$$b_\phi^{(2)} = \varepsilon^2 \cos \theta' \sin \psi(\rho \sin \theta - \sin \theta') - \varepsilon\nu\rho \sin \theta' \sin \theta(1 - \cos \psi)$$

$$\begin{aligned} b_r^{(3)} &= \varepsilon^2 \cos \theta \sin \theta' \cos \theta'(1 - \cos \psi) \\ b_\theta^{(3)} &= -\varepsilon^2 \sin \theta \sin \theta' \cos \theta'(1 - \cos \psi) \\ b_\phi^{(3)} &= 0 \end{aligned} \tag{19}$$

where $\zeta \equiv \theta' - \theta$ has been introduced.

On account of the complicated form of d given by (11) and (12), the integrations in (10) and (15) are well known to be difficult (Aleksin 1963, Sy 1981). Nevertheless, the integrals will be evaluated here as a power series in ε by writing

$$\frac{1}{d} = \frac{1}{\sqrt{2}} \frac{1}{(\lambda - \cos \psi)^{1/2}} \left[1 - \frac{\varepsilon \Delta}{2} \left(\frac{1 - \cos \psi}{\lambda - \cos \psi} \right) + \frac{3\varepsilon^2 \Delta^2}{8} \left(\frac{1 - \cos \psi}{\lambda - \cos \psi} \right)^2 \times \frac{\varepsilon^2 \rho}{2} \cos \theta \cos \theta' + \ldots \right] \tag{20}$$

where $\lambda \equiv 1 + \frac{1}{2}\delta^2$, $\Delta \equiv \cos \theta' + \rho \cos \theta$ and

$$\delta^2 \equiv (a^2 + r^2 - 2ar \cos \zeta)/R_0^2. \tag{21}$$

The method indicated here differs from that introduced by Aleksin (1963) and applied by subsequent authors (Kalyuzhnyj and Nemov 1977). This aspect is discussed below.

4. The cylindrical limit

Mathematically, toroidal geometry and cylindrical geometry are quite distinct; for example, a toroidal surface is topologically compact and has two periodicities whereas an infinite cylindrical surface is non-compact and has only one periodicity. Indeed, in certain problems on stability of an axisymmetric toroidal equilibrium (Lüst *et al* 1961), toroidal results do not tend to cylindrical results as the aspect ratio tends to infinity. Moreover, there are perturbation theories of toroidal stellarator fields (Aleksin 1963, Dobrott and Frieman 1971, Fielding and Hitchon 1980) which do not tend asymptotically to cylindrical limits as the aspect ratio tends to infinity. The pitch length in these theories are assumed to be long, in such a way that $\nu \equiv \varepsilon m/l$ vanishes in the limit of infinite aspect ratio. These classes of stellarators require large currents in the helical windings to produce a significant rotational transform.

On the other hand, from a study of helical windings on the torus (Tayler 1965, Sy 1981), it may be shown that the geodesics on the torus approach those of a straight circular cylinder as the aspect ratio tends to infinity provided $\nu^2 > 2\varepsilon$. That is, the pitch length of the helices must be sufficiently short to overcome the effects of toroidal curvature. Under these circumstances, it is then not surprising that the cylindrical limit should exist. Previous studies suggest that the cylindrical limit might be obtained under the conditions:

$$R_0 \to \infty \qquad m \to \infty \qquad k \equiv m/R_0 \text{ finite.} \tag{22}$$

It will be shown that, indeed, under these conditions, the toroidal magnetic vector potential and magnetic field both approach uniformly those of the corresponding cylinder. It is appropriate then to take this as the zeroth order in our perturbation theory.

4.1. Vector potential and magnetic field

The zeroth-order quantities may be obtained from taking $d = \sqrt{2}(\lambda - \cos\psi)^{1/2}$, where $\lambda \equiv 1 + \frac{1}{2}\delta^2$ and δ is defined by (21). Note that, as in classical potential theory, it is necessary to retain $\delta^2 \neq 0$ in d to avoid singular integrands in (10) and (15) due to vanishing denominators. On performing the ψ integrals (see appendix 1), the zeroth-order magnetic vector potential reads

$$A_\alpha^{(0)} = \frac{\mu_0 I_0}{2\pi} \int_0^{2\pi} F_\alpha \,\mathrm{d}\zeta \tag{23}$$

where one defines

$$\begin{aligned} F_r &\equiv \nu \sin l\Phi \sin \zeta \sin l\zeta K_0(x) \\ F_\theta &\equiv \nu \cos l\Phi \cos \zeta \cos l\zeta K_0(x) \\ F_\phi &\equiv \cos l\Phi \cos \zeta K_0(x) \end{aligned} \tag{24}$$

whilst the corresponding magnetic field reads

$$B_\alpha^{(0)}=\frac{\mu_0 I_0}{2\pi a}\int_0^{2\pi} G_\alpha \, d\zeta \tag{25}$$

with definitions

$$\begin{aligned}
&G_r \equiv \sin l\Phi[\varepsilon^2 m^2 \sin l\zeta \sin \zeta K_1(x)/x + \varepsilon m\nu \cos l\zeta \cos \zeta K_0(x)]\\
&G_\theta \equiv \cos l\Phi[-\varepsilon^2 m^2 \cos l\zeta (r-a\cos\zeta)K_1(x)/ax + \varepsilon m\nu \sin l\zeta \sin\zeta K_0(x)]\\
&G_\phi \equiv -\cos l\Phi \varepsilon^2 m^2 \cos l\zeta (a-r\cos\zeta)K_1(x)/ax.
\end{aligned} \tag{26}$$

In these expressions, K_0 and K_1 are modified Bessel functions of the second kind, and it has been convenient to introduce the helical variable Φ and the argument x by the definitions

$$\Phi \equiv \theta-(m/l)\phi \tag{27}$$

$$x^2 \equiv m^2\delta^2 \equiv k^2(a^2+r^2-2ar\cos\zeta) \tag{28}$$

where $k \equiv m/R_0$ may be regarded as a toroidal pitch parameter.

On performing the remaining ζ integrals in (23) and (25) (see appendix 2), the components of the magnetic vector potential read

$$\begin{aligned}
&A_r^{(0)}=-\mu_0 I_0 \sin l\Phi[K_l(ka)I_l'(kr)+(a/r)K_l'(ka)I_l(kr)]\\
&A_\theta^{(0)}=-\mu_0 I_0 r^{-1}\cos l\Phi[(a/\nu)K_l(ka)I_l(kr)+r\nu K_l'(ka)I_l'(kr)]\\
&A_\phi^{(0)}=\mu_0 I_0 \cos l\Phi K_l(ka)I_l(kr)
\end{aligned} \tag{29}$$

whilst the components of the magnetic field read

$$B_r^{(0)}=kbI_l'(kr)\sin l\Phi \qquad B_\theta^{(0)}=(lb/r)I_l(kr)\cos l\Phi \qquad B_\phi^{(0)}=-kbI_l(kr)\cos l\Phi \tag{30}$$

where $b \equiv -\mu_0 I_0 K_l'(ka)$. Consider a corresponding problem in cylindrical geometry with a harmonic current distribution given by (9) and a winding law $\theta-\theta_0=(k/l)z$. It may be verified that with identifications $dz/d\phi=-R_0$ and $\Phi=\theta-(k/l)z$, the magnetic vector potential and magnetic field obtained are identical to (29) and (30) respectively. Moreover, one has $\nabla\cdot\boldsymbol{A}=0$, $\nabla\cdot\boldsymbol{B}=0$ and $\boldsymbol{B}=\nabla\times\boldsymbol{A}$ as required.

4.2. *Solution to a stellarator problem*

The calculation of magnetic fields due to a set of thin helical current filaments on the surface of a cylinder has been considered by Aleksin (1962) and Maschke (1969). These authors calculate the field due to each individual filament and then superpose the results for the set to obtain the resultant field of stellarators and torsatrons. Due to the singular nature of the current distribution and a lesser degree of symmetry, the expressions for a single filament are rather awkward and only approximate results have been obtained. Here, it will be shown that the method of Fourier analysis discussed in § 2 leads to an exact solution in a straightforward way.

Consider the magnetic field produced by the helical current distribution (4) and a uniform axial field B_0. The superposition of harmonic magnetic fields given formally by (30) yields the general expressions

$$
\begin{aligned}
B_r &= \sum_n n\alpha b_n I'_n(n\alpha r) \sin n\Phi \\
B_\theta &= \sum_n \frac{nb_n}{r} I_n(n\alpha r) \cos n\Phi \\
B_z &= B_0 - \sum_n n\alpha b_n I_n(n\alpha r) \cos n\Phi
\end{aligned}
\tag{31}
$$

where the summation is taken over the appropriate multiples of l and $\Phi \equiv \theta - \alpha z$ is the helical variable, with $\alpha = k/l$. Note that in the case of torsatrons without an additional axial field $B_0 = \mu_0 I_0 \alpha l / 4\pi$, otherwise B_0 is arbitrary. The coefficients b_n in the above expressions are defined by

$$b_n \equiv 2\mu_0 I_0 a_n \alpha a K'_n(n\alpha a)/\pi \tag{32}$$

where $a_n \equiv \sin(\frac{1}{2}nw)/(\frac{1}{2}nw)$, as was defined by (8).

On account of helical symmetry in the cylindrical case, an exact invariant, which may be identified as the magnetic surface function, exists and it reads

$$\Psi = \tfrac{1}{2}r^2 - \sum_n (b_n/B_0) r I'_n(n\alpha r) \cos n\Phi. \tag{33}$$

The expressions (31)–(33) represent an exact solution to the cylindrical stellarator problem considered; other properties such as the position of separatrices or the rotational transform may be deduced from these results. The limiting case of $w = 0$ is the problem of thin filaments considered by the previous authors.

4.3. Conductors with finite radial dimension

From the expressions obtained for the magnetic field, it is now convenient to discuss the effect of finite radial dimensions of the helical conductors. It may be seen that on summing infinitesimally thin current layers for helical conductors extending radially from $a - \frac{1}{2}T$ to $a + \frac{1}{2}T$, the coefficients b_n in (32) must be replaced by

$$b_n = \frac{2\mu_0 I_0 a_n \alpha}{\pi a T} \int_{a-\frac{1}{2}T}^{a+\frac{1}{2}T} r^2 K'_n(n\alpha r)\, \mathrm{d}r. \tag{34}$$

Provided the total current through each conductor remains I_a, the magnetic field is still given formally by (31), but b_n is determined by (34). Analysis of the expression in (34) shows that b_n in this case differs negligibly from b_n given by (32) provided $T^2 \ll a^2 w$, which is satisfied under usual circumstances.

5. Effects of toroidal curvature

The effects of toroidal curvature may be included by retaining terms of higher order in the inverse aspect ratio. The general procedure for doing this has been indicated in § 3. Here, the effects of toroidal curvature will be discussed by including toroidal perturbations to first order in the inverse aspect ratio for the magnetic vector potential and magnetic field produced by the harmonic current distribution (9).

5.1. Toroidal perturbations

It follows from expressions derived in § 3 that the first-order perturbations to the magnetic vector potential and the magnetic field are given respectively by

$$A_\alpha^{(1)}=\frac{\mu_0 I_0}{4\pi\sqrt{2}}\int\int_0^{2\pi}\frac{\cos(l\theta'-m\phi')}{(\lambda-\cos\psi)^{1/2}}\left(a_\alpha^{(1)}-\frac{\varepsilon\Delta a_\alpha^{(0)}}{2}\frac{(1-\cos\psi)}{(\lambda-\cos\psi)}\right)\mathrm{d}\theta'\,\mathrm{d}\phi' \tag{35}$$

and

$$B_\alpha^{(1)}=\frac{\mu_0 I_0\varepsilon}{4\pi a\sqrt{2}}\int\int_0^{2\pi}\frac{\cos(l\theta'-m\phi')}{(\lambda-\cos\psi)^{3/2}}\left(b_\alpha^{(1)}-\frac{3\varepsilon\Delta b_\alpha^{(0)}}{2}\frac{(1-\cos\psi)}{(\lambda-\cos\psi)}\right)\mathrm{d}\theta'\,\mathrm{d}\phi'. \tag{36}$$

Evaluation of the integrals in these expressions is tedious but straightforward. Since it follows similar steps to those indicated in the previous section for the zeroth-order terms, the intermediate results will be omitted, but some useful relationships in carrying out the calculation are recorded in the appendices. The final expressions then may be written

$$\begin{pmatrix}A_r^{(1)}\\A_\theta^{(1)}\\A_\phi^{(1)}\end{pmatrix}=\frac{\varepsilon}{4}\sum_\pm\begin{pmatrix}a_r^\pm\sin\Omega_\pm\\a_\theta^\pm\cos\Omega_\pm\\a_\phi^\pm\cos\Omega_\pm\end{pmatrix} \tag{37}$$

and

$$\begin{pmatrix}B_r^{(1)}\\B_\theta^{(1)}\\B_\phi^{(1)}\end{pmatrix}=\frac{\varepsilon}{2}\sum_\pm\begin{pmatrix}b_r^\pm\sin\Omega_\pm\\b_\theta^\pm\cos\Omega_\pm\\b_\phi^\pm\cos\Omega_\pm\end{pmatrix} \tag{38}$$

where $\Omega_\pm\equiv(l\pm1)\Phi\pm\tau$ and $\tau\equiv m\phi/l$. Note that the angular variables $\Omega_\pm$ are separated into the variable of helical symmetry $\Phi\equiv\theta-(m/l)\phi$ and the toroidal variable τ which breaks this symmetry. Moreover, it is seen that toroidal curvature creates $l\pm1$ sidebands. The coefficients $a_\alpha^\pm$ and $b_\alpha^\pm$ are functions of r only, independent of the angular variables and they are given by

$$\begin{aligned}a_r^\pm&=-\mu_0 I_0\nu(P_{l\pm1}+\rho P_l-W_{l\pm1}-\rho W_l)\\a_\theta^\pm&=-\mu_0 I_0\nu(Q_{l\pm1}+\rho Q_l-V_{l\pm1}-\rho V_l)\\a_\phi^\pm&=\mu_0 I_0(O_{l\pm1}-\rho O_l+U_{l\pm1}+\rho U_l)\end{aligned} \tag{39}$$

and

$$\begin{aligned}b_r^\pm&=(\mu_0 I_0/a)[\varepsilon^2m^2(P_{l\pm1}+\rho P_l+R_{l\pm1}-\rho R_l)+\varepsilon m\nu(2O_l-3Q_{l\pm1}\\&\qquad-3\rho Q_l+V_{l\pm1}+\rho V_l)\pm O_l\mp U_l]\\b_\theta^\pm&=(\mu_0 I_0/a)[\varepsilon^2m^2(Q_{l\pm1}+\rho Q_l+S_{l\pm1}-\rho S_l-\rho O_{l\pm1}-\rho^2O_l)\\&\qquad+\varepsilon m\nu(W_{l\pm1}+\rho W_l-3P_{l\pm1}-3\rho P_l\mp2O_l\pm O_{l\pm1})+O_l-U_l]\\b_\phi^\pm&=(\mu_0 I_0/a)[\varepsilon^2m^2\nu(\rho Q_{l\pm1}-O_{l\pm1}-T_{l\pm1})\pm2\varepsilon m(O_{l\pm1}-\rho O_l)+\nu(O_{l\pm1}-U_{l\pm1}).\end{aligned} \tag{40}$$

The quantities $O_n, P_n, Q_n, \ldots, W_n$ are integrals over Bessel functions of argument x defined by (28); they are written down and evaluated in appendix 2. It may now be seen that a toroidal stellarator magnetic field can be regarded as composed of two parts: the first part is identical to a cylindrical stellarator field, possessing helical symmetry and independent of τ, whilst the smaller second part arises from toroidal curvature and depends sinusoidally on τ, breaking the helical symmetry.

5.2. Magnetic field line equations

The effect of toroidal geometry on the magnetic structure may be studied from a suitable system of equations for the magnetic field lines. On writing the axisymmetric toroidal magnetic field as $B_{\phi 0} - B_0 R_0/R$, a new toroidal variable t may be introduced by

$$\mathrm{d}\tau/\mathrm{d}t = B_\phi/B_0. \tag{41}$$

As shown above, the zeroth order field lines, unperturbed by toroidal curvature, have an invariant Ψ_0 given by

$$\Psi_0 \equiv \tfrac{1}{2}\rho^2 - (b/B_0)\rho I_l'(l\nu\rho)\cos l\Phi \tag{42}$$

where $b/B_0 \equiv -\nu\varepsilon_I K_l'(l\nu)$ and $\varepsilon_I \equiv \mu_0 I_0/B_0 a$. The field line equations for the toroidal case may then be written

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{1}{\rho}\frac{\partial\Psi_0}{\partial\Phi} + \frac{1}{\nu B_0}(\varepsilon\rho\cos\theta B_r^{(0)} + B_r^{(1)}) \\ \frac{\mathrm{d}\Phi}{\mathrm{d}t} &= -\frac{1}{\rho}\frac{\partial\Psi_0}{\partial\rho} + \frac{1}{\nu\rho B_0}(\varepsilon\rho\cos\theta B_\theta^{(0)} + B_\theta^{(1)} - \nu\rho B_\phi^{(1)}) \end{aligned} \tag{43}$$

where $\theta = \Phi + \tau$ and $B_\alpha^{(1)}$ are sinusoidal functions of τ. To the accuracy required, one may identify here the independent variables τ and t.

From (43), it can be seen that the unperturbed system is a conservative system with Hamiltonian Ψ_0 and conjugate variables $\frac{1}{2}\rho^2$ and Φ. Hence the whole system with the symmetry-breaking toroidal perturbations may then be regarded as a conservative Hamiltonian system under small non-stationary perturbations.

6. The toroidal magnetic structure

The effect of toroidal curvature on the magnetic structure is now examined by applying the method of averaging (Bogolyubov and Mitropolskii 1961) to the system of equations (43) to obtain approximate magnetic surfaces. A meridonal section of the torus can be regarded as a Poincaré section for the 'motion' of a magnetic field line circulating around the torus according to the equations of motion (43). From asymptotic arguments, it is reasonable to anticipate that for sufficiently small toroidal curvature, magnetic surfaces would exist and that they are slightly distorted from those of the unperturbed system.

For cases considered in this paper, a particular field line experiences several helical field periods on going once around the torus before returning to a reference meridional plane. The toroidal perturbations are sinusoidal functions of t, which goes through several periods on one transit around the torus. Hence to obtain sections of magnetic surfaces at one meridional plane, averages over t may be used. The average of a function f over t, which has a period 2π, may be defined by

$$\langle f\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f\,\mathrm{d}t. \tag{44}$$

On introducing generalised coordinates x^i, the system of equations (43) can be written in a general form

$$\mathrm{d}x^i/\mathrm{d}t = f^i \equiv \langle f^i\rangle + \tilde{f}^i \qquad i=1,2 \tag{45}$$

where $\tilde{f}^i$ is the oscillatory part. In using the method of averaging, new variables ξ^i, sometimes called averaged variables, are introduced by

$$x^i = \xi^i + \hat{f}^i(\xi^i, t) \tag{46}$$

where $\hat{f}^i$ are oscillatory functions defined by

$$\hat{f}^i = \int^t \tilde{f}(\xi^i, t')\,\mathrm{d}t' - \left\langle \int^t \tilde{f}(\xi^i, t')\,\mathrm{d}t' \right\rangle. \tag{47}$$

It can be shown (e.g. Morozov and Solov'ev 1966) that for small $f^i \sim \mathrm{O}(\varepsilon)$, the averaged equations are given by

$$\mathrm{d}\xi^i/\mathrm{d}t = \langle f^i\rangle + \langle \hat{f}^j(\partial f^i/\partial \xi^j)\rangle. \tag{48}$$

Within the present approximations, the last term in (48) is of the order ε^2 and may be neglected. From (43) and (45), it can be deduced that an approximate invariant is given by $\Psi_0(\xi^i) = \text{constant}$. From an inversion of (46), it may be seen that the true magnetic surfaces including terms of order ε are given by

$$\Psi(x^i, t) \equiv \Psi_0(x^i) - \hat{f}^j(x^i, t)\partial\Psi_0/\partial x^j = \text{constant}. \tag{49}$$

It may be verified directly by differentiation that

$$\mathrm{d}\Psi/\mathrm{d}t = \partial\Psi/\partial t + (\mathrm{d}x^i/\mathrm{d}t)(\partial\Psi/\partial x^i) = \mathrm{O}(\varepsilon^2).$$

To provide the simplest illustration of the effect of toroidal curvature, consider only toroidal variation of the superimposed axisymmetric toroidal magnetic field. In this case, the magnetic surfaces may be written

$$\Psi(\rho, \theta, \phi) = \tfrac{1}{2}\rho^2 - (b/B_0)\rho I_l'(l\nu\rho)[\cos(l\theta - m\phi) + l\varepsilon\rho \sin\theta \sin(l\theta - m\phi)]. \tag{50}$$

For cases, where $\varepsilon m < 1$, asymptotic forms of the Bessel functions in (50) may be used to reduce the expression to a simple approximate form:

$$\Psi(\rho, \theta, \phi) = \tfrac{1}{2}\rho^2 - (\varepsilon_l/2\nu l)\rho^l[\cos(l\theta - m\phi) + l\varepsilon\rho \sin\theta \sin(l\theta - m\phi)]. \tag{51}$$

Examples are given for an $l=2$ case in figure 2 and an $l=3$ case in figure 3. An indication that toroidal curvature breaks the helical symmetry may be seen from the fact that, allowing for a suitable poloidal rotation, the magnetic surfaces no longer preserve their form as the toroidal angle is advanced.

Unlike other systems of magnetic field line equations in the literature (e.g. Kalyuzhnyj and Nemov 1977), the right-hand sides of (43) are small near the separatrix region and hence we have been able to use (49) to obtain magnetic surfaces in the

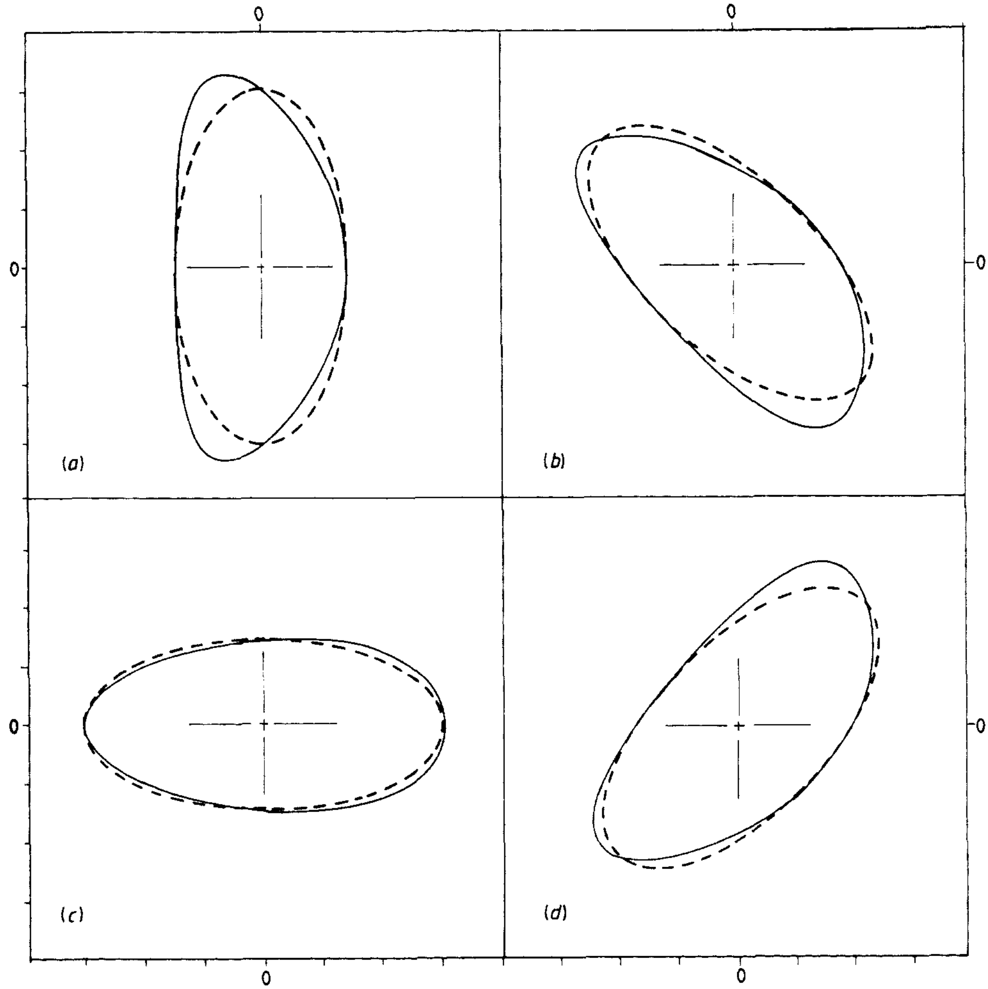

Figure 2. Effect of toroidal curvature on $l=2$ magnetic surfaces. Full contours denote sections of outer magnetic surfaces for the toroidal case, whilst the broken contours denote those for the corresponding cylindrical case, for various values of $m\phi$: (*a*) 0°, (*b*) 90°, (*c*) 180°, (*d*) 270°. The vacuum chamber has inverse aspect ratio 0.3, major axis to to the left of diagram and minor axis located by a cross.

outer regions. It has been seen that symmetry-breaking toroidal perturbations cause shifts in the location of separatrices determined by the singular points of (43) and they give rise to distortions of the magnetic surfaces. However, these perturbations have been calculated in this paper only for a toroidal stellarator with classical helical windings. Numerical studies (Blamey *et al* 1982) suggest that adverse effects of toroidal geometry might be alleviated by suitable modulations of the classical windings. The present study indicates that indeed it appears fruitful to minimise the non-stationary terms on the right-hand side of (43) to reduce the effects of toroidal curvature. It is therefore not worthwhile at this stage to present numerical examples of magnetic surfaces for extensive ranges of parameters merely for the cases of classical windings.

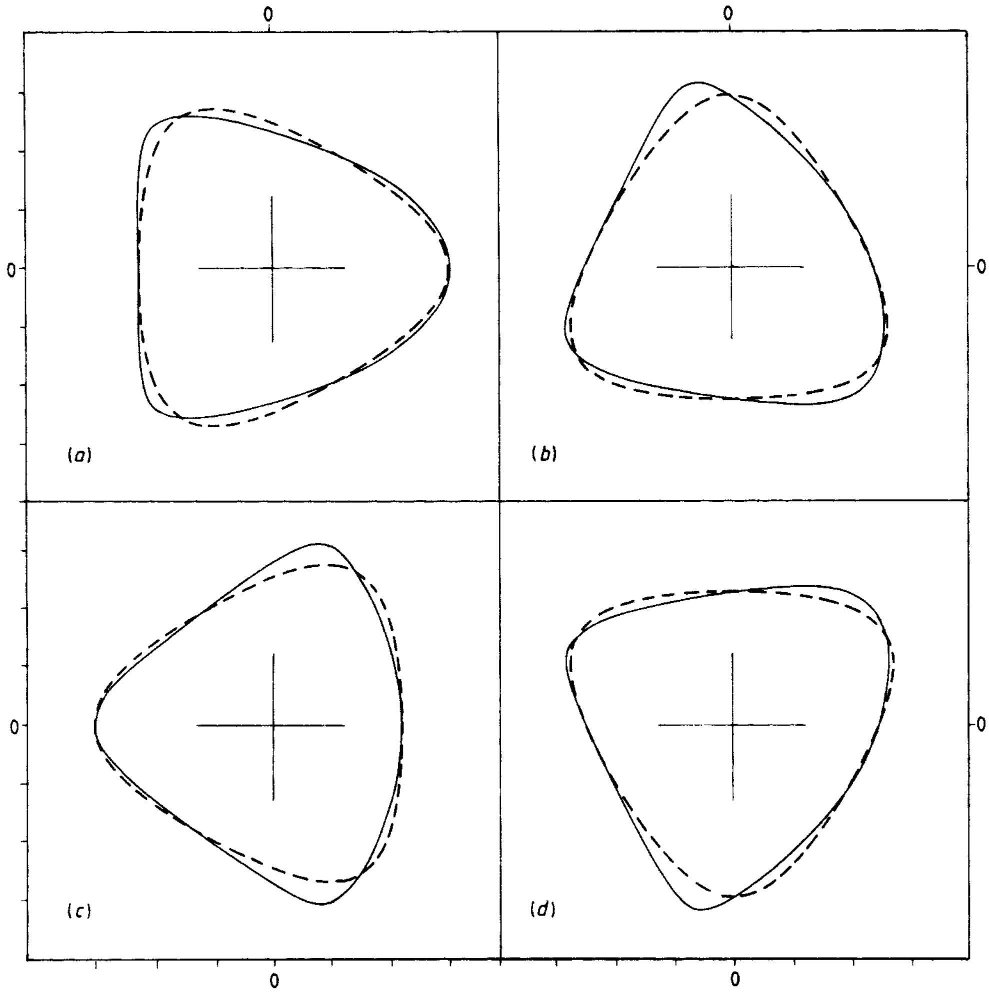

Figure 3. Effect of toroidal curvature on $l = 3$ magnetic surfaces; parameters and description are same as those for figure 2.

7. Concluding remarks

In this paper a new perturbation theory of the toroidal stellarator magnetic field has been presented. The zeroth-order theory is identical to that of the cylindrical stellarator and may be obtained in the limit as the number of toroidal periods m and major radius R_0 both tend to infinity in such a way that $k \equiv m/R_0$ remains finite. In ordering schemes where m is not ordered as ε^{-1}, such as those considered by Dobrott and Frieman (1971) and Fielding and Hitchon (1980), where $m = O(\varepsilon^{-2/3})$, the cylindrical limit is not obtained as ε tends to zero.

The toroidal perturbations have been shown to be purely geometric in the sense that they vanish in the limit of zero toroidal curvature. These perturbations are seen to break the helical symmetry of the cylindrical limit and render the magnetic structure truly three dimensional, requiring for description a toroidal variable in addition to a helical symmetry variable and a radial coordinate. The toroidal perturbations calculated are more general than those obtained by previous authors, particularly in the following respects. (i) Unlike those of Kalyuzhnyj and Nemov (1977) and others, the expressions are applicable to all regions inside the torus, including the separatrix region. (ii) No ordering assumptions, such as those of Greene and Johnson (1961) and Dobrott and Frieman (1971) have been made about the helical field amplitude as compared to the superimposed toroidal field. (iii) The long pitch length assumption: $\nu \equiv \varepsilon m/l \ll 1$ (Aleksin 1963) has not been made.

In using the Biot–Savart law to calculate magnetic fields which satisfy explicitly actual boundary conditions on the torus, such as those associated with finite-size helical conductors, it has been shown that analysis of the current distribution into a series of harmonic distributions can lead to considerable computational simplification, as well as insights into harmonic generation due to finite dimensions of the helical conductors. Exact expressions for the magnetic surfaces of the cylindrical case have been derived for current distributions with finite-size helical conductors.

The effect of symmetry-breaking toroidal perturbations on the cylindrical system has been formulated as a problem of non-stationary perturbations in a conservative Hamiltonian dynamical system. From approximate analytical methods, toroidal curvature has been shown to lead to distortions of magnetic surfaces, as expected from numerical studies (Blamey *et al* 1982). It is beyond the scope of the present paper to give a fuller mathematical discussion of the system of equations (43); the paper by Nehoroshev (1977) contains some recent mathematical advances relevant to this problem. In conclusion, we note that modulations of the helical windings on the torus may reduce the perturbative terms in (43) and thus lead to a partial restoration of helical symmetry. This aspect appears worthy of further investigation.

Acknowledgments

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Appendix 1. Approximations and expansions of the ψ integrals

The basic ψ integral of interest has an exact representation in terms of generalised associated Legendre functions:

$$\int_0^{2\pi} \frac{\cos m\psi \, d\psi}{(\lambda - \cos\psi)^{n+1/2}} = \frac{2\sqrt{2}}{(\lambda^2-1)^{n/2}} \frac{(-1)^n 2^{2n} n!}{(2n)!} Q^n_{m-1/2}(\lambda). \tag{A1.1}$$

This result can be proved by applying differentiation formulae to the well known $n = 0$ case (Hobson 1931).

When $m \gg 1$, $\lambda = 1 + \frac{1}{2}\delta^2$ and $\delta \ll 1$, $Q^n_{m-1/2}(\lambda)$ may be expanded in a series of modified Bessel function $K_n(x)$ (Robin 1958)

$$Q^n_{m-\frac{1}{2}}(\lambda) = \frac{x^n(-1)^n}{[2(\lambda-1)]^{n/2}} \left[K_n(x) + \frac{\lambda-1}{2}\left(\frac{x}{6}K_{n-3}(x) - K_{n-2}(x) + \frac{1}{2x}K_{n-1}(x) + \frac{n}{2}K_n(x)\right) + O[(\lambda-1)^2]\right] \tag{A1.2}$$

where $x = m[2(\lambda - 1)]^{1/2} = m\delta$. It follows that to the leading term

$$\int_0^{2\pi} \frac{\cos m\psi \, d\psi}{(\lambda - \cos\psi)^{n+1/2}} = \frac{2\sqrt{2}(2m)^{2n} n!}{x^n (2n)!} K_n(x). \tag{A1.3}$$

For most applications in the present paper only low values of n are required. In particular, the following list, which may be deduced from further calculations using (A1.3), has been found to be useful.

$$\int_0^{2\pi} \frac{\cos m\psi \cos\psi \, d\psi}{(\lambda - \cos\psi)^{1/2}} = 2\sqrt{2} K_0(x) \tag{A1.4}$$

$$\int_0^{2\pi} \frac{\sin m\psi \sin\psi \, d\psi}{(\lambda - \cos\psi)^{1/2}} = 2\sqrt{2}\,\delta K_1(x) \tag{A1.5}$$

$$\int_0^{2\pi} \frac{\cos m\psi (1 - \cos\psi) \, d\psi}{(\lambda - \cos\psi)^{3/2}} = 2\sqrt{2}\,(K_0(x) - x K_1(x)) \tag{A1.6}$$

$$\int_0^{2\pi} \frac{\cos m\psi \cos\psi \, d\psi}{(\lambda - \cos\psi)^{3/2}} = 4\sqrt{2}\, m^2 K_1(x)/x \tag{A1.7}$$

$$\int_0^{2\pi} \frac{\sin m\psi \sin\psi \, d\psi}{(\lambda - \cos\psi)^{3/2}} = 4\sqrt{2}\, m K_0(x) \tag{A1.8}$$

$$\int_0^{2\pi} \frac{\cos m\psi (1 - \cos\psi) \, d\psi}{(\lambda - \cos\psi)^{5/2}} = \frac{4\sqrt{2}}{3} m^2 (K_1(x)/x - K_0(x)). \tag{A1.9}$$

The orders, with respect to the inverse aspect ratio ε, of these integrals may be seen from $m = O(\varepsilon^{-1})$, $\delta = O(\varepsilon)$ and $x = O(1)$.

Appendix 2. Evaluation of the ζ integrals

For $x^2 \equiv k^2(a^2 + r^2 - 2ar\cos\zeta)$, $K_0(x)$ may be expanded into a Fourier series by an addition theorem (Watson 1966)

$$K_0(x) = \sum_{n=-\infty}^{\infty} K_n(ka) I_n(kr) \cos n\zeta. \tag{A2.1}$$

Partial differentiations with respect to the arguments ka, kr and ζ can be used to show

$$k(a - r\cos\zeta) K_1(x)/x = -\sum_{n=-\infty}^{\infty} K'_n(ka) I_n(kr) \cos n\zeta \tag{A2.2}$$

$$k(r - a\cos\zeta) K_1(x)/x = -\sum_{n=-\infty}^{\infty} K_n(ka) I'_n(kr) \cos n\zeta \tag{A2.3}$$

$$k^2 ar \sin\zeta K_1(x)/x = \sum_{n=-\infty}^{\infty} n K_n(ka) I_n(kr) \sin n\zeta. \tag{A2.4}$$

The ζ integrals occurring in § 5 are defined as follows:

$$O_n \equiv \frac{1}{2\pi} \int_0^{2\pi} \cos n\zeta K_0(x) \, d\zeta \tag{A2.5}$$

$$P_n \equiv \frac{1}{2\pi}\int_0^{2\pi} \sin\zeta \sin n\zeta K_0(x)\,\mathrm{d}\zeta \tag{A2.6}$$

$$Q_n \equiv \frac{1}{2\pi_0} \cos\zeta \cos n\zeta K_0(x)\,\mathrm{d}\zeta \tag{A2.7}$$

$$R_n \equiv \frac{1}{2\pi}\int_0^{2\pi} \sin\zeta \sin n\zeta (K_1(x)/x)\,\mathrm{d}\zeta \tag{A2.8}$$

$$S_n \equiv \frac{1}{2\pi}\int_0^{2\pi} (\cos\zeta - \rho) \cos n\zeta (K_1(x)/x)\,\mathrm{d}\zeta \tag{A2.9}$$

$$T_n \equiv \frac{1}{2\pi}\int_0^{2\pi} (\rho\cos\zeta - 1) \cos n\zeta (K_1(x)/x)\,\mathrm{d}\zeta \tag{A2.10}$$

$$U_n \equiv \frac{1}{2\pi}\int_0^{2\pi} \cos n\zeta (xK_1(x))\,\mathrm{d}\zeta \tag{A2.11}$$

$$V_n \equiv \frac{1}{2\pi}\int_0^{2\pi} \cos\zeta \cos n\zeta (xK_1(x))\,\mathrm{d}\zeta \tag{A2.12}$$

$$W_n \equiv \frac{1}{2\pi}\int_0^{2\pi} \sin\zeta \sin n\zeta (xK_1(x))\,\mathrm{d}\zeta. \tag{A2.13}$$

These integrals can be evaluated on applying the following relationships, where K_n has argument ka and I_n has kr,

$$\begin{aligned} &O_n = K_n I_n \qquad P_n = \tfrac{1}{2}(O_{n-1} - O_{n+1}) \qquad Q_n = \tfrac{1}{2}(O_{n-1} + O_{n+1}) \\ &R_n = -(Q_n + K'_n I'_n)/n \qquad S_n = K_n I'_n/ka \qquad T_n = K'_n I_n/ka \\ &U_n = k^2 ar P_n/n \qquad V_n = \tfrac{1}{2}(U_{n-1} + U_n) \qquad W_n = \tfrac{1}{2}(U_{n-1} - U_n) \end{aligned} \tag{A2.14}$$

where $P_0 = R_0 = U_0 = W_0 = 0$ and $V_0 = U_1$.

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